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MINIMAL HYPER-HAMILTON LACEABLE GRAPHS

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Abstract—A Hamilton laceable graph G on n nodes is defined to be *hyper-Hamilton laceable* if either

(1) G is equitable and if v is any node of G, $G - v$ is Hamilton laceable, or
(2) G is nearly equitable and if v is any node in the larger color set, $G - v$ is Hamilton laceable.

The minimum number of edges needed for G to be hyper-Hamilton laceable in the case where $n = 2k > 4$ is shown to be $3k$. In the case where $n = 2k + 1 > 3$, the minimum number of edges, E_n, is shown to satisfy the inequality

$$3k \leq E_n \leq 4k - 1.$$

1. INTRODUCTION

A graph G is *Hamilton connected* if for every pair x, y of distinct nodes of G, there exists an $x-y$ *Hamilton path*, i.e., an $x - y$ path containing all the nodes of G. A bipartite graph is said to be *equitable* if both color sets have the same cardinality. If the difference in the cardinalities is one, then the bipartite graph is called *nearly equitable*. A bipartite graph which is neither equitable nor nearly equitable is called *skewed*. A bipartite graph is not Hamilton connected. In light of this fact, Simmons [1] gives the following definition.

A bipartite graph G is Hamilton laceable if either

(1) *G is equitable and for each pair of nodes x and y of opposite color, there exists an $x - y$ Hamilton path, or*
(2) *G is nearly equitable and for each pair of distinct nodes x and y in the larger color set, there exists an $x - y$ Hamilton path.*

The hypercube Q_n is an example of a Hamilton laceable graph. For more information on Hamilton laceable graphs, see [1–4].

In [3], we defined hyper-Hamilton laceable graphs as follows.

A Hamilton laceable graph G is hyper-Hamilton laceable if either

(1) *G is equitable and if v is any node of G, $G - v$ is Hamilton laceable, or*
(2) *G is nearly equitable and if v is any node in the larger color set, $G - v$ is Hamilton laceable.*

The graphs C_4 and $K_{3,2}$ are examples of hyper-Hamilton laceable graphs.

We then presented [3] the following lemma, theorem, and corollary.

LEMMA A. *$C_{2m} \times P_n$, m, $n \geq 2$, is hyper-Hamilton laceable.*

An n-dimensional mesh M is *even* if the order of M is even; otherwise it is *odd*.

THEOREM A. *Even n-dimensional meshes, $n \geq 3$, are hyper-Hamilton laceable.*

COROLLARY A. *Q_n is hyper-Hamilton laceable.*

An equitable Hamilton laceable graph is Hamiltonian and therefore contains C_{2m} as a spanning subgraph. Hence, in light of Lemma A, we have the following theorems.

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THEOREM 1. *If G is equitable hyper-Hamilton laceable of order ≥ 4 and H is traceable bipartite, then $G \times H$ is hyper-Hamilton laceable.*

THEOREM 2. *If G and H are hyper-Hamilton laceable and at least one is equitable with order ≥ 4, then $G \times H$ is hyper-Hamilton laceable.*

Let G be an equitable or nearly equitable bipartite graph, and let $J(G)$ be the graph that results from joining a node u to either color set if G is equitable, or to the larger color set if G is nearly equitable. It is shown in [2] that if G is Hamilton laceable, then so is $J(G)$. We present a slightly stronger result.

THEOREM 3. *If G is an equitable Hamilton laceable graph, then $J(G)$ is hyper-Hamilton laceable.*

PROOF. Since G is equitable, $J(G)$ is nearly equitable. We must show that $J(G) - v$ is Hamilton laceable, where v is any node in the larger color set. Clearly, the newly added node u is in the larger color set. If $v = u$, then $J(G) - v = G$ which is Hamilton laceable by hypothesis. Suppose $v \neq u$. Let x and y be oppositely colored nodes such that y and v have the same color. To obtain an $x - y$ Hamilton path where $y = u$, take an $x - v$ Hamilton path in G and label the node prior to v on this path w. The desired Hamilton path is then $\{x - w\} \cup w\,y$. To obtain an $x - y$ Hamilton path where $y \neq u$, take an $x - y$ Hamilton path in G which has v on it. Since the neighborhood of v is a subset of the neighborhood of u, we can replace v by u, thereby yielding an $x - y$ Hamilton path for $J(G) - v$. ∎

COROLLARY 1. *$J(Q_n)$ is hyper-Hamilton laceable.*

THEOREM 4. *If G is a nearly equitable hyper-Hamilton laceable graph, then so is $J(G)$.*

PROOF. Since G is a nearly equitable hyper-Hamilton laceable graph, it is a nearly equitable Hamilton laceable graph. Thus, from the statement above, $J(G)$ is an equitable Hamilton laceable graph. We need to show $J(G) - v$ is Hamilton laceable. If $v = u$, then $J(G) - v = G$ which is obviously Hamilton laceable. Suppose now that $v \neq u$. There are two cases to consider.

CASE 1. u and v are in the same color set.

In this case, the endnodes x and y of the desired Hamilton path are in the larger color set of G. Therefore, there exists an $x - y$ Hamilton path in G which contains v. Since the neighborhood of v is a subset of the neighborhood of u, we can replace v by u, thereby yielding an $x - y$ Hamilton path for $J(G) - v$.

CASE 2. u and v are in different color sets.

Then v is in the larger color set of G. Since G is nearly equitable hyper-Hamilton laceable, $G - v$ is equitable Hamilton laceable. Since $J(G) - v = J(G - v)$, it follows from Theorem 3 that $J(G) - v$ is Hamilton laceable. ∎

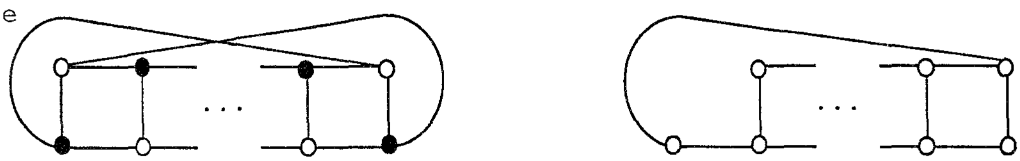
2. MINIMAL HYPER-HAMILTON LACEABLE GRAPHS

THEOREM 5. *The minimum number of edges in a hyper-Hamilton laceable graph on $n = 2k$ nodes, $k > 2$, is $3k$.*

PROOF. In [4], Simmons shows that for nearly equitable Hamilton laceable graphs, every node in the smaller color set has degree at least three. For equitable hyper-Hamilton laceable graphs, any node v can be deleted to produce a nearly equitable Hamilton laceable graph. Therefore, every node of a hyper-Hamilton laceable graph must have degree at least three. Thus, we must have at least $3k$ edges.

This bound of $3k$ edges in a hyper-Hamilton laceable graph is realized when k is even, by the graph $C_k \times P_2$, which is hyper-Hamilton laceable in light of Lemma A. For k odd, we have the graph shown in Figure 1a. Simmons shows this graph minus edge e to be Hamilton laceable. The addition of this edge does not change the laceability of the graph. If we delete a node from this graph, we obtain the graph of Figure 1b, which Simmons proves is Hamilton laceable with the fewest number of edges. ∎

Since every node of an equitable hyper-Hamilton laceable graph on more than four nodes must have degree at least three, it follows that, with the exception of C_4, no even two-dimensional



(a) (b)

Figure 1.

meshes are hyper-Hamilton laceable. If a node of degree two is deleted from an odd two-dimensional mesh, the resulting graph contains a node of degree greater than two which is adjacent to two nodes of degree two. In [4], Simmons proves that an equitable Hamilton laceable graph cannot have such a node. Thus, no odd two-dimensional mesh is hyper-Hamilton laceable.

THEOREM 6. *The minimum number of edges, E_n, in a hyper-Hamilton laceable graph on $n = 2k + 1$ nodes, $k > 1$, satisfies*

$$3k \le E_n \le 4k - 1.$$

PROOF. The minimum number of edges in a Hamilton laceable graph on $2k + 1$ nodes is $3k$, (see [4]). Since a hyper-Hamilton laceable graph is Hamilton laceable, it must have at least $3k$ edges. The upper bound is realizable by $J(G)$, where G is one of the graphs of Figure 2 depending on whether k is odd or even. Simmons proves that the graphs of Figure 2 are Hamilton laceable. It follows from Theorem 3 that $J(G)$ is hyper-Hamilton laceable when G is either of these graphs. ∎

(a) (b)

Figure 2.

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