ON TWO-LEGGED CATERPILLARS WHICH SPAN HYPERCUBES

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ABSTRACT

A caterpillar is a tree which becomes a path when its endnodes are removed. A two-legged caterpillar has maximum degree four. A strictly two-legged caterpillar has degree set $\{1,2,4\}$. While a characterization of the two-legged caterpillars on 2^n nodes which span a hypercube Q_n is at present unknown, we present classes of strictly two-legged caterpillars which span hypercubes. A tight upper bound is given for the number of nodes of degree four in strictly two-legged caterpillars which span a hypercube.

I. Introduction

A caterpillar T is a tree which becomes a path when its endnodes are removed. The path is called the spine of T. A k-legged caterpillar T satisfies $\Delta T = k + 2$. When the degree set of a k-legged caterpillar is $\{1,2,k+2\}$, we call T strictly k-legged. Figure 1 depicts a strictly 2-legged caterpillar.

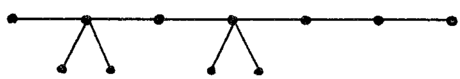

Figure 1. A strictly 2-legged caterpillar.

The hypercube Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$. The order of Q_n is 2^n. The hypercube is bipartite, i.e., 2-colorable, and as its two color sets have the same number of nodes, it is equitable. It follows that spanning trees of Q_n are necessarily equitable. A characterization of the caterpillars on 2^n nodes which span Q_n is currently unknown, though Havel and Liebl proved the following in [2] .

Theorem A. Equitable 1-legged caterpillars on 2^n nodes span Q_n. ▯

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II. Strictly 2-legged caterpillars

We turn our attention to strictly 2-legged caterpillars which span Q_n. The following lemma establishes criteria for equitability. We omit the proof.

Lemma 1. A strictly 2-legged caterpillar is equitable if and only if its spine is of even order and it has the same number of nodes of degree four in each of its two colors. $\square$

As a consequence of Lemma 1, strictly 2-legged caterpillars must have an even number of nodes of degree four. We begin, therefore, with strictly 2-legged caterpillars on 2^n nodes with exactly two nodes of degree four. The following lemma will be needed to establish our first theorem.

Lemma 1a. Given non-adjacent edges $e,e' \in E(Q_n)$, there exists a division of Q_n into two node-disjoint copies H and H' of Q_{n-1} such that $e \in E(H)$ and $e' \in E(H')$.

Proof. Let $e = xy$ and $e' = uv$. Assume without loss of generality that $d(x,u) = \min\left[d(x,u),d(x,v),d(y,u),d(y,v)\right]$. There are two cases.

Case 1. $d(x,u) = 1$. Let D be the unique cut-set of 2^{n-1} edges such that $xu \in D$ and $Q_n - D$ consists of two copies of Q_{n-1}. Clearly $e,e' \notin D$ and e and e' belong to different components of $Q_n - D$.

Case 2. $d(x,u) = k \geq 2$. Let L be an x-u geodesic path. Let w be the vertex of L adjacent to x. Let D be the cutset, as defined above, such that $xw \in D$. Let H and H' be the components of $Q_n - D$ containing x and w respectively. Then $xy \in E(H)$. Now since L is a geodesic path and $xw \in D$, none of the remaining edges of L belong to D. Then since $w \in V(H')$, so is u, from which it follows that $uv \in E(H')$. $\square$

Theorem 1. Given an equitable strictly 2-legged caterpillar T on 2^n nodes $(n \geq 4)$ with exactly two nodes of degree four, then T spans Q_n.

Proof. Denote by $U(2^n-4;k)$ the unicyclic graph formed by attaching two pendant edges to each of two nodes x and y of a cycle of order $2^n - 4$, with $d(x,y) = k$, as shown in Figure 2 for $n = 4$ and $k = 3$.

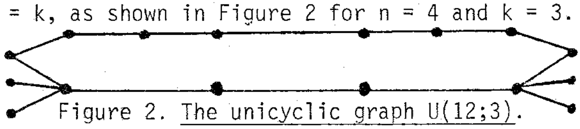

Figure 2. The unicyclic graph U(12;3).

To prove the theorem, it suffices to show that for $n \geq 4$, $U(2^n-4;k)$ spans Q_n when k is an odd integer (by Lemma 1) and $1 \leq k \leq 2^{n-1} - 3$, since any caterpillar of Theorem 1 can then be obtained by deleting a cycle edge of $U(2^n-4;k)$. We proceed by induction on n. For $n = 4$, Figure 3 exhibits an embedding of $U(12;1)$. By deleting edges yc and yd and adding uc and ud, we obtain an embedding of $U(12;3)$. If we then delete xa and xb and add va and vb, we obtain an embedding of $U(12;5)$.

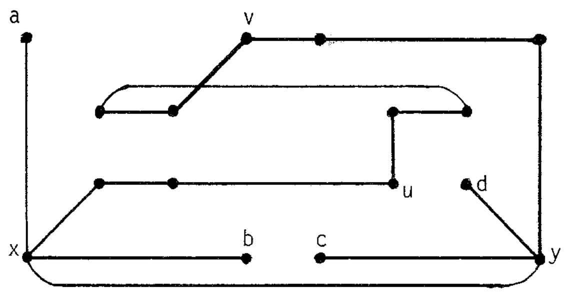

Figure 3. <u>An embedding of $U(12;1)$ in Q_4</u>.

Now assume that the theorem is true up to n - 1 and consider $U(2^n-4;k)$. There are two cases.

<u>Case 1</u>. $k \leq 2^{n-1} - 5$. Consider Q_n as two copies of Q_{n-1}, called Q^L_{n-1} and Q^U_{n-1}, with "vertical" edges connecting corresponding nodes in both copies. Let $j = \min(k, 2^{n-1} - 4 - k)$. By the inductive hypothesis, we can embed $U(2^{n-1}-4;j)$ in Q^L_{n-1}. Let u and v be consecutive nodes on the path of length $2^{n-1} - 4 - k$ of the cycle of $U(2^{n-1}-4;j)$ and let u' and v' be their corresponding nodes in Q^U_{n-1}. Let L be a hamiltonian u'-v' path for Q^U_{n-1}. This is possible since hypercubes are hamilton laceable [1]. Then $[U(2^{n-1}-4;j) - uv] \cup uu' \cup vv' \cup L$ spans Q_n and is isomorphic to $U(2^n-4;k)$.

<u>Case 2</u>. $k = 2^{n-1} - 3$. Again consider Q_n as two copies of Q_{n-1}. Embed $U(2^{n-1}-4;2^{n-2}-3)$ in Q^L_{n-1}. Let e and e' be cycle edges on opposite paths between the two nodes of degree 4. By Lemma 1a, there exists a division of Q^L_{n-1} into two copies, G and H, of Q_{n-2} such that $e \in E(G)$ and $e' \in E(H)$. Let G' and H' be the corresponding copies of G and H in Q^U_{n-1}. Denoting e by xy and e' by uv, let M be a hamiltonian x'-y' path for G' and let N be a hamiltonian u'-v' path for H'. Then

$[U(2^{n-1}-4;2^{n-2}-3) - xy - uv] \cup xx' \cup yy' \cup uu' \cup vv' \cup M \cup N$ spans Q_n

and is isomorphic to $U(2^n-4;2^{n-1}-3)$. ▯

We use the usual caterpillar code in which a sequence of non-negative integers denotes the number of non-spinal endnodes adjacent to the spinal nodes. Figure 4 exhibits an embedding in Q_4 of (0,2,2,0,0,2,2,0).

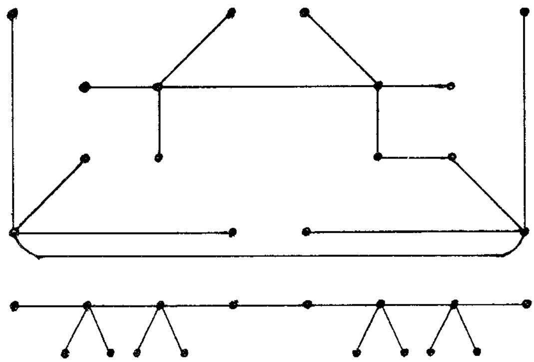

Figure 4. An embedding of (0,2,2,0,0,2,2,0) in Q_4.

Theorem 2. The strictly 2-legged caterpillar with code (0,2,2,0,0,2,2,...,0,0,2,2,0) spans Q_n for $n \geq 4$.

Proof. By induction on n. Figure 4 shows that the theorem is true when n = 4. Assume it is true for n = k. Then let x be the first spinal node (with code 0) of (0,2,2,...,0,0,2,2,0) which, by the inductive hypothesis spans Q_k. Now take another copy of Q_k with the same caterpillar embedded in it, and let x' in the second copy correspond to x. Upon adding "vertical" edge xx', we obtain the required spanning caterpillar of Q_{k+1}, thereby completing the proof of the theorem. ▯

III. The maximum number of nodes of degree four

We ask for the maximum possible number of nodes of degree four in a strictly 2-legged caterpillar which spans Q_n. When n is odd, this number cannot exceed $(2^n - 2)/3$, while when n is even, it cannot exceed $(2^n - 4)/3$, since by Lemma 1, the number of nodes of degree four is even. The next theorems show that these upper bounds are, in fact, achievable.

Theorem 3. Let n be an odd integer such that $n \geq 5$. Then there exists a strictly 2-legged caterpillar T_n which spans Q_n, and T_n contains $(2^n - 2)/3$ nodes of degree four.

Proof. Let U_n be the unicyclic graph on 2^n nodes with a cycle of order $(2^n + 4)/3$ such that, with the exception of two adjacent cycle-nodes, each cycle-node has two pendant edges attached. Clearly, to prove Theorem 3, it suffices to show that U_n spans Q_n for odd $n \geq 5$. We proceed by induction on n. When $n = 5$, Figure 5 shows an embedding of U_5 in Q_5. Note that $U_5 - ab$ in Figure 5 is the caterpillar (0,2,2,2,2,2,2,2,2,2,2,0).

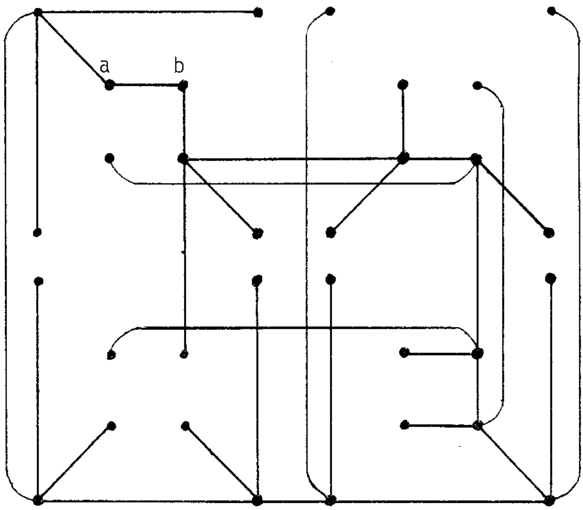

Figure 5. An embedding of U_5 in Q_5.

Now assume the theorem is true for $n = k$ (where k is odd), i.e., U_k embeds in Q_k and we must show that U_{k+2} embeds in Q_{k+2}. Consider Q_{k+2} as four copies of Q_k, denoted H_i, $i = 1,2,3,4$ which may be thought of as Q_k upper left, upper right, lower left, and lower right, respectively. Edges between H_1 and H_2 and between H_3 and H_4 are called "horizontal", while those between H_1 and H_3 and between H_2 and H_4 are "vertical". By the inductive hypothesis, we embed U_k^i into H_i for each $i = 1,2,3,4$. For each i, let a_i, b_i, c_i, and d_i be consecutive cycle-nodes of degrees 4,2, 2, and 4, respectively, such that the subgraphs of Q_{k+2} induced by the a_i, b_i, c_i, and d_i are 4-cycles.

We construct U_{k+2} from the embedded U_k^i as follows. Delete edges $b_1c_1, c_1d_1, a_2b_2, b_3c_3, c_3d_3$ and a_4b_4. Then add horizontal edges c_1c_2 and c_3c_4 and vertical edges a_2a_4, b_1b_3, c_2c_4, and d_1d_3. Finally, select two consecutive cycle-nodes u_1 and v_1 of U_k^1 with corresponding cycle-nodes u_2 and v_2 of U_k^2 and add horizontal edges u_1u_2 and v_1v_2 while deleting u_1v_1 and u_2v_2. The resulting copy of U_{k+2} spans Q_{k+2}. ∎

Theorem 4. Let n be an even integer such that $n \geq 6$. Then there exists a strictly 2-legged caterpillar which spans Q_n and contains $(2^n - 4)/3$ nodes of degree four.

Proof. Consider Q_n as two node-disjoint copies, H_1 and H_2 of Q_{n-1}. Since $n - 1$ is odd, by Theorem 3, H_i contains a copy of U^i_{n-1}, with nodes a_i, b_i, c_i, and d_i (for $i = 1$ and 2) as previously defined. We obtain the desired spanning caterpillar of Q_n from the U^i_{n-1} by adding the edge d_1d_2 and deleting edges c_1d_1 and c_2d_2. As each of the U^i_{n-1} has $(2^{n-1} - 2)/3$ nodes of degree four, the caterpillar we constructed has twice this number of nodes of degree four, proving the theorem. ▯

Note that Theorem 4 yields the correct number for $n = 4$, though the method of proof is inapplicable in this case.

References

[1] F.Harary and M.Lewinter, Hypercubes and other recursively defined Hamilton laceable graphs. Congressus Numerantium, to appear (1988).

[2] I.Havel and P.Liebel, One-legged caterpillars span hypercubes. J.Graph Theory 10, 1986, 69-77.